Hamiltonian Evolution Equations – Where They Come From, What They Are Good For

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"Some people talk in their sleep. Lecturers talk while other people sleep."

(Albert Camus)

Examples of Hamiltonian Evolution Equations

- Euler Equations (& "quantization" ➔ vortex dynamics)
- Vlasov Eq. (& its "quantization" ➔ point-particle mechanics) Maxwell-Vlasov, etc.
- Non-linear Schrödinger– and Hartree Eqs. (non-focusing & *focusing*: e.g., boson stars, structure formation; soliton dynamics, KAM theorems for soliton dynamics (?), etc.)
- Hartree-Fock- and Bogoliubov-Hartree-Fock Eqs. (e.g., collapse of white dwarfs – collapse profile; atomic and molecular physics, superconductivity, etc.)
- Maxwell-axion dynamics (➔ growth of cosmic magnetic fields)
- Coupled particle-wave dynamics, with wave medium = Bose gas, or electromagnetic field – Cherenkov radiation, Hamiltonian friction, etc.; (Gang's talks).

Contents

1. Mean-field limits, etc.
2. Boson Stars
3. Chandrasekhar Limit of stellar collapse
4. Concluding remarks

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Mathematical themes underlying this lecture

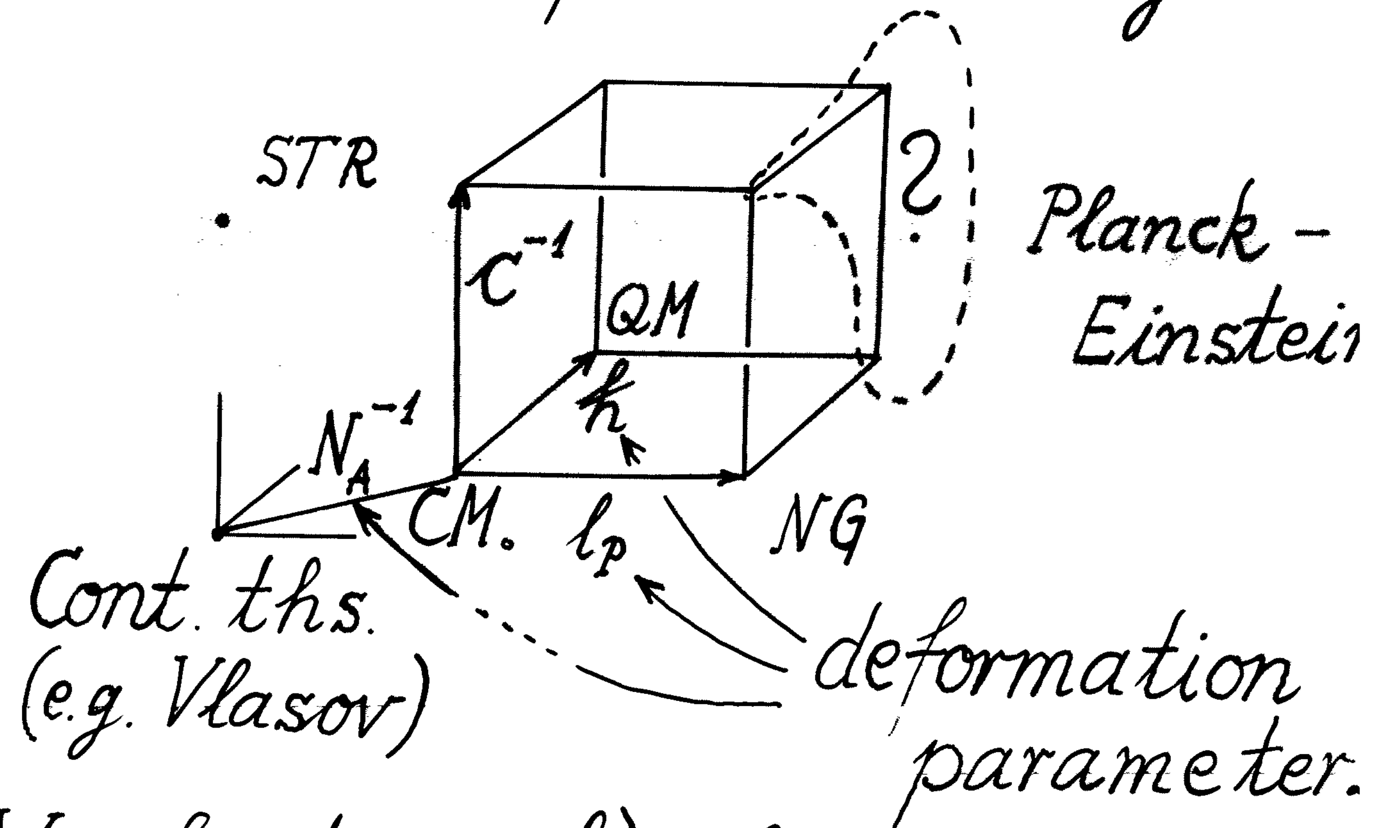
(1) Quantization $\longleftrightarrow$
deformations of assoc. algebras (Poisson br. $\rightarrow$ commutators; Fock space)

(2) Semi-class. analysis for systs. w. ∞ many degs. of freedom (Egorov thm. ...)

(3) NL Hamiltonian evolution equations, soliton dyn.

1. Atomism as quantization

Reflection on main changes in "Weltbild" of 20^{th} Cent. Physics.



(Verification of) old paradigm:

Atomism $\longleftrightarrow$ N_A^{-1}

Three revolutions:

QM $\longleftrightarrow$ $\hbar$

STR $\longleftrightarrow$ c^{-1}

GR $\longleftrightarrow$ l_P

?

21^{st} Cent.: ? $\longleftrightarrow$ α', a_-

Atomism

Über das Boltzmann'sche Prinzip und einige unmittelbar aus demselben fliessende Folgerungen.

Die Thermodynamik beruht ~~bekanntlich~~ auf zwei Prinzipien, dem Energieprinzip (auch 1. Hauptsatz genannt) und dem Prinzip von der Nichtumkehrbarkeit des Naturgeschehens (auch 2. Hauptsatz genannt). letzteres ~~ist Prinzip von sagt aus,~~ Der Inhalt des letzteren Prinzip lässt sich ~~in der Planck'schen~~ nach Planck so aussprechen

Alle Natur-Wissenschaft ist auf die Voraussetzung der vollständigen ~~lückenlosen~~ kausalen Verknüpfung jeglichen Geschehens begründet. Nehmen wir an ~~Wenn~~ Galilei hätte bei seinen ~~Fall und~~ Pendelversuchen gefunden ~~hätte~~, dass dasselbe Pendel so schwingt, dass die Dauer einer Schwingung in unregelmässiger Weise wechselt, Nehmen wir ferner an, ~~ohne~~ dass dieser Wechsel mit dem Wechsel ~~irgend~~ irgendwelcher anderer beobachtbarer Verhältnisse nicht hätte in Verbindung gebracht werden können. Dann wäre es Galilei unmöglich gewesen ~~wohl kaum eingefallen~~ seine Beobachtungen zu einem Gesetze zu vereinigen. Hätten alle uns zugänglichen Erscheinungen einen derart unregelmässigen Charakter, wie wir dies in dem soeben fingierten Falle uns vorgestellt haben, so wären die Menschen gewiss nie auf naturwissenschaftliche Bestrebungen verfallen.

Welchen Charakter müssten die Erscheinungen haben, damit Wissenschaft möglich sei? Darauf möchte man zuerst etwa folgendermassen antworten: Bringen wir ein System in einen bestimmten Zustand, so ist, falls das System von anderen Systemen – etwa durch grosse räumliche Entfernung getrennt ist, so ist ~~der~~ zeitliche Ablauf der Zustände dieses Systems vollkommen bestimmt; d. h. bringen wir ~~zwei~~ beliebig viele gleichbeschaffene isolierte Systeme in genau denselben Zustand und überlassen wir diese Systeme sich selbst, so ist für alle diese Systeme der zeitliche Ablauf der Erscheinungen genau derselbe.

A. Einstein

1911

1.1. Newtonian Mechanics as "quantization" of Vlasov Mech

"Stellar dust" descr. as class continuous medium; states given by mass density

$$M f(x,p)\, dx\, dp$$

on $\mathbb{R}^3_{position} \times \mathbb{R}^3_{velocity}$, with

$$\int f(x,p)\, dx\, dp = \nu$$

(ν : # moles of dust)

Time-dependence of $f(x,p)$ given by Vlasov Equation.

This is a model of matter as a continuous medium!

Vlasov Eq.

$$\partial_t f_t(x,p) = -\frac{1}{M}\left(p\cdot\nabla_x f_t\right)(x,p) + \left(\nabla V_{eff}[f_t]\cdot\nabla_p f_t\right)(x,p),$$

$$V_{eff}(x) := V(x) + \int dy\, \phi(x-y) \times \int dp\, f_t(y,p)$$

ϕ : reg. Newtonian pot.

& Neunzert

Braun-Hepp: Vlasov is mean-field limit of $n = \nu N_A$ point particles w. mass $m = \frac{M}{N_A}$, 2-body pot. $\frac{1}{N_A}\phi$, as $N_A \to \infty$.

Vlasov dynamics is Hamiltonian dynamics:

$$f(x,p) = \overline{\alpha(x,p)} \cdot \alpha(x,p),$$

$$\alpha \in \Gamma = H^1(\mathbb{R}^6), \ \{\alpha^\#, \alpha^\#\} = 0,$$

$$\{\alpha(x,p), \overline{\alpha(x',p')}\} = -i\,\delta(x-x')\,\delta(p-p'),$$

$$\mathcal{H}_V(\bar{\alpha},\alpha) = i\iint dx\,dp\ \bar{\alpha}\left[\frac{1}{M} p\cdot\nabla_x - \nabla V\cdot\nabla_p\right]\alpha$$

$$-i\iint dx\,dp\,\bar{\alpha}\left[\iint dy\,dr\,\nabla\phi(x-y)|\alpha(y,r)|^2\right]\cdot\nabla_p\alpha$$

Hamiltonian Eqs. of motion,

$$(1)\ \dot{\alpha}_t(x,p) = \{\mathcal{H}_V, \alpha_t(x,p)\},\ \dot{\bar{\alpha}}_t(x,p) = \cdots$$

$\Rightarrow$ Vlasov Eqs. for $f_t(x,p)$!

Wick quantization

$$\alpha(x,p) \to a_N(x,p)\ ,\ \overline{\alpha(x,p)} \to a_N^*(x,p)$$

$$[a_N^\#, a_N^\#] = 0\ ,\quad (N \equiv N_A)$$

$$[a_N(x,p), a_N^*(x',p')] = \frac{1}{N}\delta(x-x')\delta(p-p')$$

$$\sim \frac{i}{N}\{\alpha(x,p), \overline{\alpha(x',p')}\}\ (\text{Dirac})$$

a_N, a_N^* act on Fock space

$$\mathcal{F}_V := \bigoplus_{n=0}^{\infty} \mathcal{F}_V^{(n)},$$

$$\mathcal{F}_V^{(0)} = \mathbb{C}|0\rangle\ ,\ |0\rangle \text{ vacuum}$$

$$a_N(x,p)|0\rangle = 0,\ \forall x,p.$$

$$\mathcal{F}_V^{(n)} := \left\langle \int\cdots\int \varphi_n(x_1,p_1,\cdots,x_n,p_n) \prod_{i=1}^{n} a_N^*(x_i,p_i)|0\rangle \right\rangle$$

$f_n := |\varphi_n|^2 =$ symm. density on

n - point-part. phase space $\Gamma^{(n)}$

Hamilton op. : $\hat{\mathcal{H}}_V := :\mathcal{H}_V(a_N^*, a_N):$

Schrödinger Eq. :

$$iN^{-1}\partial_t \Psi_t = \hat{\mathcal{H}}_V \Psi_t, \quad \Psi_t \in \mathcal{F}_V$$

$\Leftrightarrow$ Liouville's Eq. of motion for symm. n-particle densities, $f_n = \overline{\varphi_n} \cdot \varphi_n$, on $\Gamma^{(n)}$, 2-body potential $\frac{1}{N}\phi$, $n = 0, 1, 2, \cdots$.

Apparently, atomistic Newtonian mech. of point-part.

= quantization of continuum theory given by Vlasov eq.

$\|$

(B-H) "classical limit" of Newtonian mech.

Rephrasing Braun-Hepp

$$\Psi \equiv \Psi^{(n)}(\alpha) := cst. \int \cdots \int \prod_{i=1}^{n} \alpha(x_i, p_i) a_N^*(x_i, p_i) |0\rangle$$

For $n \simeq \nu N$,

$$e^{-itN\hat{\mathcal{H}}_\nu} \Psi^{(n)}(\alpha) \simeq \Psi^{(n)}(\alpha_t) + O\left(\frac{1}{N}\right),$$

where α_t solves (1), i.e., $f_t = \nu \bar{\alpha}_t \cdot \alpha_t$ solves Vlasov, with $\int f_t = \nu$.

More precise statement in form of a <u>Egorov Theorem</u>.

Alas, description of stars in terms of Vlasov Eq. leads to instabilities

$\longrightarrow$ Quantize ($\hbar$)!

1.2 Quant. gases as "2^{nd} quantization" of Hartree mechanics

Replace $f(x,p) = \overline{\alpha(x,p)} \cdot \alpha(x,p)$

by

(2) $f_\hbar(x,p) := \frac{1}{(2\pi)^3} \int dy\, e^{-iy \cdot p}\, \overline{\psi\left(x - \frac{\hbar y}{2}\right)} \cdot \psi\left(x + \frac{\hbar y}{2}\right)$

$f_\hbar$ is Wigner trsf. of ψ

Dynamics of ψ:

(3) $i\hbar \partial_t \psi_t = \left[-\frac{\hbar^2}{2m}\Delta + V\right]\psi_t + \left[|\psi_t|^2 * \phi\right]\psi_t$

Hartree Equation

If solution ψ_t of (3) is plugged into (2) then

$$\lim_{\hbar \searrow 0} f_{\hbar,t}(x,p)$$

solves Vlasov Eq. (1);

($\nearrow$ Narnhofer – Sewell)

Hartree is Hamiltonian Eq. of motion on phase space $\Gamma = H^1(\mathbb{R}^3)$; Poisson brackets

$$\{\psi^\#, \psi^\#\} = 0, \quad \{\psi(x), \overline{\psi(y)}\} = i\delta(x-y)$$

Hamilton functional

$$\mathcal{H}_H(\bar{\psi}, \psi) := \hbar^{-1} \int dx\, \overline{\psi(x)} \left[-\frac{\hbar^2}{2m} \Delta_x + V \right] \psi(x)$$

$$+ \frac{\hbar^{-1}}{2} \int dx \int dy\, |\psi(x)|^2 \phi(x-y) |\psi(y)|^2$$

$$(3') \quad \dot{\psi}_t(x) = \{\mathcal{H}_H, \psi_t(x)\}, \quad \dot{\bar{\psi}}_t(x) = \cdots$$

<u>Continuum</u> (field) theory of a <u>quantum</u> gas

"Second" quantize:

$$\psi(x) \to \hat{\psi}_N(x), \quad \overline{\psi(x)} \to \hat{\psi}_N^*(x), \text{ w.}$$

$$[\hat{\psi}_N^\#, \hat{\psi}_N^\#] = 0, \quad [\hat{\psi}_N(x), \hat{\psi}_N^*(y)] = \frac{1}{N}\delta(x-y)$$

Fock space

$$\mathcal{F}_H = \bigoplus_{n=0}^{\infty} \mathcal{F}_H^{(n)}, \quad \mathcal{F}_H^{(0)} = \mathbb{C}|0\rangle,$$

$$\hat{\psi}_N(x)|0\rangle = 0, \ \forall x$$

$$\mathcal{F}_H^{(n)} := \left\langle \int\cdots\int \varphi_n(x_1,\dots,x_n) \prod \hat{\psi}_N^*(x_i)|0\rangle \right\rangle$$

Many-body Hamiltonian

$$\hat{\mathcal{H}}_N := :\mathcal{H}_H(\hat{\psi}_N^*, \hat{\psi}_N): \tag{4}$$

$$iN^{-1}\partial_t \Psi_t = \hat{\mathcal{H}}_N \Psi_t$$

is Schrödinger Eq. for Bose gas of $n = 0, 1, 2, 3, \cdots$ quantum point particles, "atoms", with 2-body potential $\frac{1}{N}\phi$.

On n-particle subspace $\mathcal{F}_H^{(n)}$, (4) is equivalent to

$$H^{(n)} = \sum_{j=1}^{n}\left[-\frac{\hbar^2}{2m}\Delta_j + V(x_j)\right] + \frac{1}{N}\sum_{1\leq i<j\leq n}\phi(x_i - x_j)$$

$$(4') \qquad i\hbar\partial_t \varphi_t^{(n)}(x_1,\dots,x_n) = \left[H^{(n)}\varphi_t^{(n)}\right](x_1,\dots,x_n)$$

$$\begin{tikzcd}
\boxed{\text{QM many-body theory}} \arrow[r, shift left] \arrow[r, shift right, leftarrow, swap] \arrow[r, phantom, "\hbar", yshift=-12pt] \arrow[d, shift right, "\frac{1}{N}"'] & \boxed{\text{Class. mech of pt. part.}} \arrow[d, shift right, "\frac{1}{N}"] \\
\boxed{\text{Hartree dynamics}} \arrow[u, shift right] \arrow[r, shift right] & \boxed{\text{Vlasov dynamics}} \arrow[u, shift right] \arrow[l, shift right, "\hbar"']
\end{tikzcd}$$

Atomism $\longleftrightarrow$ "2^{nd}" Quantization

2. A Egorov Theorem

Continuum theory of quantum gases: "Observables" ~ gauge-inv. functions on Γ

$$A(a^{(p)}) = \int \cdots \int \prod_{1}^{p} \overline{\psi(x_i)} dx_i \, a^{(p)}(\underline{x}; \underline{y}) \prod_{1}^{p} \psi(y_j) \, dy_j$$

Time evolution given by flow,

$\Phi_t : \overset{(-)}{\psi} \to \overset{(-)}{\psi}_t$, where $\overset{(-)}{\psi}_t$ solves Hartree Eq. (3), (3')

$$A(a^{(p)}) \to A_t(a^{(p)}) := A(a^{(p)}) \circ \Phi_t$$

Many-body theory of quantum gases: $\psi \mapsto \hat{\psi}_N$, $\overline{\psi} \mapsto \hat{\psi}_N^*$, with CCR; $A(a^{(p)}) \mapsto \hat{A}_N(a^{(p)})$,

$$\hat{A}_N(a^{(p)}) = \int \cdots \int \prod_1^p \hat{\psi}_N^*(x_i)\,dx_i \; a^{(p)}(\underline{x},\underline{y}) \prod_1^p \hat{\psi}_N(y_j)\,dy_j$$

gauge-inv., $\hat{\psi}_N^{\#} \longmapsto e^{\pm i\theta} \hat{\psi}_N^{\#}$,

i.e., preserves particle #.

Time evolution (Heisenberg)

$$\hat{A}_N(a^{(p)}) \longrightarrow$$

$$\hat{A}_{N,t}(a^{(p)}) = e^{itN\hat{\mathcal{H}}_N} \hat{A}_N(a^{(p)}) e^{-itN\hat{\mathcal{H}}_N}$$

Conservation laws:

Gauge-inv. $\longleftrightarrow$ $\mathcal{N}(\bar{\psi},\psi) = \int |\psi|^2\,dx$

$$\downarrow$$

$\hat{\mathcal{N}}$ = part. #

Time-transl.-inv. $\longleftrightarrow$ $\mathcal{H}_H(\bar{\psi},\psi)$

$$\downarrow$$

$$\hat{\mathcal{H}}_N$$

Egorov Theorem (new!)

For $n \lesssim \nu N$, $(\nu < \infty)$,

$$\hat{A}_{N,t}(a^{(p)})\Big|_{\mathcal{J}^{(n)}} \underset{N\to\infty}{=} \left(\widehat{A(a^{(p)})\circ\Phi_t}\right)_N\Big|_{\mathcal{J}^{(n)}} + o(1)$$

Idea of proof:

$\hat{A}_{N,t}(a^{(p)})$ in "interaction pict.";

expand in Lie-Schwinger series; $<$ tree terms / loop terms

Using conservation laws,

$$|\text{loop terms}| \sim O\left(\tfrac{1}{N}\right);$$

$$\underbrace{\sum \text{tree terms}}_{\text{abs. conv., } |t| \text{ small, unif. in } N.} = \left(\widehat{A(a^{(p)})\circ\Phi_t}\right)_N$$

Related story for Fermions

$\psi \to \hat{\psi}_N$, $\bar{\psi} \to \hat{\psi}^*_N$, with

$$[\hat{\psi}^\#_N, \hat{\psi}^\#_N]_+ = 0, \quad [\hat{\psi}_N(x), \hat{\psi}^*_N(y)]_+ = \frac{1}{N}\delta(x-y),$$

$$[A,B]_+ := AB + BA.$$

$$\mathcal{H}_H \to \hat{\mathcal{H}}^f_N, \quad A(a^{(p)}) \to \hat{A}^f_N(a^{(p)}),$$

$a^{(p)}(x_1,\dots,x_p; y_1,\dots,y_p)$ tot. anti-symm in x's & in y's.

There's again a Egorov-type theorem. But "continuum theory" given by

? Hartree-Fock Eq. for $n \sim \nu N$ orbitals.

1. Phys. systems & models

- Bosonic atoms in magn. trap BEC
- Bosonic dark matter, e.g. axions – Newtonian 2-body ints. → giant bosonic molecules: $N \sim 10^{30}$

QM descript.: Hilbert space of pure state vects

$$\mathcal{H}^{(N)} = L^2(\mathbb{R}^{3N})_{sym.} \simeq L^2(\mathbb{R}^3)^{\otimes_s N}$$

Dynamics generated by Hamiltonian

\# atoms: N

strength of 2-body interaction: $\varkappa$

$N \to \infty$, $\varkappa \to 0$, $\varkappa \cdot N \equiv \nu = cst$

($\leftrightarrow$ "$\hbar \to 0$")

Useful to explore:

(a) BEC in gases of bosonic atoms (^{87}Rb, ^{23}Na) with very weak repulsive 2-body int.

$\varkappa$ very small

(Gross - Pitaevski, ...

Lieb, Seiringer, Yngvason,

(b) If 2-body int. is attractive, and $\kappa N > \nu_*$, i.e. $N > N_c$, collapse to **bound** clusters of "atoms" $\sim$ class. particles!

Examples: ^7Li, bosonic stars, ...

(c) mean field limit + $m_{boson} \to \infty$: dark matter, structure formation, boson stars

(HE) = eqs. of motion of Hamiltonian syst. with ∞ many degs. of freedom

Phase space $\Gamma := H_{1/2/1}(\mathbb{R}^3)$,

complex coos. $u(x), \bar{u}(x)$,

Poisson brackets

$$\{u^\#(x), u^\#(y)\} = 0, \{u(x), \bar{u}(y)\} = i\delta(x-y)$$

Classical Hamilton function

$$\mathcal{H}_\nu(\bar{u}, u) := \int dx \; \bar{u}(x)\,(hu)(x) - $$

$$- \frac{\nu}{2} \iint dx\, dy \, |u(x)|^2 \Phi(x-y) |u(y)|$$

$$\dot{u}_t = \{\mathcal{H}_\nu, u\}_t, \quad \dot{\bar{u}}_t = \{\mathcal{H}_\nu, \bar{u}\}_t$$

$\Longleftrightarrow$ (HE)

(pseudo-relat.) Hartree eq.

Conservation laws

- $\mathcal{H}_\nu(\bar{u},u) \longleftrightarrow$ time transl. inv.

  If ext. pot. $V \equiv$ const. :

- $\mathcal{P}(\bar{u},u) := \int d^3x\, \bar{u}(x)\left(-i\nabla u\right)(x)$

  $\longleftrightarrow$ space transl. inv.

- $\mathcal{N}(\bar{u},u) := \int d^3x\, |u(x)|^2$

  $\longleftrightarrow$ gauge invariance

Let $\quad h := \sqrt{-\Delta + m^2} + V$

2. Stability of light boson stars (L-Y)

$$H_{\kappa N} := \sum_{j=1}^{N} \left\{ \sqrt{p_j^2+m^2} - m \right\} - \kappa \sum_{1\le i<j\le N} |x_i - x_j|^{-1}$$

$$E_\kappa^Q(N) := \inf \operatorname{spec} H_{\kappa N}$$

$$\mathcal{E}_\kappa^H(u) := \left\langle u, \left\{ \sqrt{p^2+m^2} - m \right\} u \right\rangle - \frac{\kappa}{2} \iint \frac{|u(x)|^2\,|u(y)|^2}{|x-y|}\,dx\,dy$$

$$E_\kappa^H(N) := \inf \left\{ \mathcal{E}_\kappa^H(u) \,\middle|\, u \in H_{1/2},\ \mathcal{N}(u,\bar{u}) = N \right\}$$

$$E_\kappa^H(N) = -\infty \quad \text{if } N > \frac{\nu_c}{\kappa},$$

some $\nu_c > 0$.

<u>Theorem</u> (L-Y)

$\nu := \kappa N < \nu_c$ fixed. Then

$$\lim_{N\to\infty} E_\kappa^Q(N) \big/ E_\kappa^H(N) = 1$$

If $\nu > \nu_c$ $\lim_{N\to\infty} E_\kappa^Q(N) = -\infty$.

$$\left(\lim_{\kappa\to 0} N_c^Q(\kappa) \big/ \kappa^{-1}\nu_c = 1\right)$$

same Chandrasekhar limits!

If $\nu < \nu_c$ $\mathcal{E}_\kappa^H(u)$ has a minimizer u_*, with $|u_*|^2$ symm.-decreasing

Conjecture on excit. spect

spec$\left(H_{\frac{\nu}{N}, N}\right)$: $\times \quad \times \quad \times \; \times \cdots [\!\!\!\text{———}$

$\uparrow$ $E^{Q}_{\frac{\nu}{N}}(N)$ $\qquad\qquad \Sigma_N$

eigenvalue spacing $O(N^{\cdots}$

as $N \to \infty$; lowest excita-

tion energies from

$$\sqrt{-\Delta+m^2}\,\chi_i - \nu \Phi * |u_*|^2 \chi_i = (\varepsilon_i + m)\chi_i$$

$$\chi_0 = u_*\,, \quad \frac{\varepsilon_0}{2} \underset{N\to\infty}{\sim} \frac{E^{Q}_{\frac{\nu}{N}}(N)}{N}$$

$$\nu < \nu_c$$

$$\Sigma_N = E^{Q}_{\frac{\nu}{N}}(N-1) \sim E^{Q}_{\frac{\nu}{N}}(N) - \varepsilon_0$$

3. Gravitational collapse of heavy boson stars

$$i\partial_t u_t = \sqrt{p^2+m^2}\, u_t - \nu\left(\frac{e^{-M|x|}}{|x|} * |u_t|^2\right) u_t \qquad (HE)$$

$\mathcal{E}^H_\nu(u) \equiv \mathcal{H}(u,\bar{u})$ as in Sects. 1 & 2.

$$E_0 := \mathcal{E}^H_\nu(u_0) + m\,\|u_0\|_2^2\,,$$

u_0 some initial cond. for (HE) at $t=0$; (Lenzmann: local well-posedness)

<u>Theorem</u> (F-L)

$$m \geq 0, \ M \geq 0$$

u_0 spherically symm.,

$u_0 \in H_2(\mathbb{R}^3)$, $|x|^2 u_0(x) \in L^2(\mathbb{R}^3)$,

If $E_0 < 0$ then solu.

$u_t(x)$ of (HE) blows up

at time $t = T < \infty$, in

the sense that

$$\lim_{t \nearrow T} \|u_t\|_{H_{1/2}} = \infty$$

Idea of proof inspired

by methods in QM.

Let $$Q := x \cdot \sqrt{p^2+m^2}\, x$$

$$\mathcal{Q}(t) := \langle u_t, Q\, u_t \rangle > 0$$

Show that, for $t < T$,

\# $0 < \mathcal{Q}(t) \leq 2E_0 t^2 + at + b$,

a, b finite consts.

Since $E_0 < 0$, R.S. has a zero in t at $t = t_*$.

Then $0 < T \leq t_*$

\# $\Leftrightarrow$ $\ddot{\mathcal{Q}}(t) \leq 4E_0$ (virial th.)

\- estimates on commutators!

1) $$\frac{d}{dt}\langle u_t, Q u_t\rangle \leq 2\langle u_t, A u_t\rangle + C\|u_0\|_2^4,$$

$$A := -\frac{i}{2}(x\cdot\nabla + \nabla\cdot x)$$

2) $$\frac{d}{dt}\langle u_t, A u_t\rangle \leq 2E_0$$

Results extend to (HE) with ext. potential: Add

$$V(x)u_t(x).$$

Grav. collapse if

$$E_0 < -\frac{1}{2}\|V + x\cdot\nabla V\|_\infty \|u_0\|_2^2$$

$\underline{E_0 > 0?}$

Formation of solitons (?)

(T. Tao)

4. Gravitating Quantum Systems

G: Newton's grav. constant

m_{nuc}: mass of typical nucl., charge Ze, in stellar matter

Grav. attraction betw. 2 nuclei at distance R

$$-\frac{G m_{nuc}^2}{R}$$

$$\frac{1}{N} := \frac{G m_{nuc}^2}{\hbar c} \sim 10^{-36} \ll 1 \; !$$

Stellar systems:

n electrons of mass m, el. charge $-e$

$\frac{n}{Z}$ nuclei of charge Ze
electric neutrality
Particles confined to "ball"
of radius R;
Fermi momentum $p \simeq \frac{n^{1/3}}{R}$
(work w. units s.t. $\hbar = c = 1$)
If $p \sim m$ relat. kinematics
for electrons; $m_{nuc} \gg m$
Estimate of grav. energy

$$E(n,R) \simeq \left[n\sqrt{p^2+m^2} + \frac{n m_{nuc}}{Z} - \frac{1}{2}\left(\frac{n}{Z}\right)^2 \frac{G m_{nuc}^2}{R} \right]_{p \simeq \frac{n^{1/3}}{R}}$$

$$\min_R E(n,R) = \begin{cases} E_0(n) > -\infty, & n < N_{Ch} \\ -\infty, & n > N_{Ch}, \end{cases}$$

$$N^{3/2} \propto N_{Ch} \sim 10^{57}, \quad M_{Ch} \sim \frac{m_{nuc}}{Z} N_{Ch} > M_{\odot}$$

$$\frac{1}{N} \propto \kappa := \frac{G m_{nuc}^2}{Z^2} \ll 1$$

→ Describe electron gas in mean-field limit, with relat. kinematics; distr. of nuclei ~ distr. of electr. charge neutrality

→ Pseudo-relat. Hartree-Fock Eq

$$\text{(5)} \quad \begin{aligned} i\partial_t \psi_k = \sqrt{-\Delta + m^2}\,\psi_k &- \sum_{\ell=1}^{n} \left(\frac{\kappa}{|x|} * |\psi_\ell|^2\right)\psi_k \\ &+ \sum_{\ell=1}^{n} \left(\frac{\kappa}{|x|} * \bar{\psi}_\ell \cdot \psi_k\right)\psi_\ell \end{aligned}$$

with $\langle \psi_k, \psi_l \rangle = \delta_{kl}$, $k, l = 1, \dots, n$

$n < \infty$.

Last term on R.S. of (5) is "exchange term" – subleading as $n \to \infty$ → neglect it!

(5) is Hamiltonian Eq. of motion on $\Gamma^{(n)} := \left(H^{1/2}(\mathbb{R}^3)\right)^{\times n}$

$$\Psi := (\psi_k)_{k=1}^{n} \in \Gamma^{(n)}$$

Hamilton functional

$$\mathcal{H}_H(\overline{\Psi}, \Psi) := \sum_{k=1}^{n} \langle \psi_k, \sqrt{-\Delta + m^2}\, \psi_k \rangle - \frac{\kappa}{2} \iint \frac{\rho_\Psi(x)\, \rho_\Psi(y)}{|x-y|}\, dx\, dy \qquad (6)$$

$$\rho_\Psi(x) := \sum_{k=1}^{n} |\psi_k(x)|^2$$

(in (6), exchange term has been neglected)

Conservation laws (Noether)

$U(n)$ gauge-inv. $\leftrightarrow$ $\langle \psi_k, \psi_l \rangle$

time-transl.-inv. $\leftrightarrow$ $\mathcal{H}_H(\bar{\Psi}, \Psi)$

$H^s := H^s(\mathbb{R}^3)^{\times n}$, with norm

$$\|\Psi\|_s := \Big(\sum_{k=1}^{n} \|\psi_k\|^2_{H^s(\mathbb{R}^3)} \Big)^{1/2},$$

$$s \geq \frac{1}{2}$$

5. Results for Hartree (-Fock)

Hartree - Fock Eq. (5) $\Rightarrow$

(5') $$\Psi(t) = U_0(t)\Psi(0) + \int_0^t U_0(t-s)\,\varkappa F(\Psi(s))\,d.$$

$$U_0(t) = e^{-it\sqrt{-\Delta+m^2}},$$

$$F_k(\Psi) := -\sum_{\ell=1}^{n}\left(\frac{1}{|x|} * |\psi_\ell|^2\right)\psi_k + \sum_{\ell=1}^{n}\left(\frac{1}{|x|} * \overline{\psi_\ell}\,\psi_k\right)\psi_\ell$$

For any n, for $s \geqslant \frac{1}{2}$,

$$F : H^s \longrightarrow H^s$$

is locally Lipschitz.

$$\Downarrow$$

1. Local well-posedness of (5) for arb. $\Psi(0) \in H^s$, $s \geqslant \frac{1}{2}$

T: max. time of existence of solution, $\Psi(t)$, of (5)

T is independent of $s \geq \frac{1}{2}$

($\nearrow$ E. Lenzmann, thesis)

2. Global existence ($T = \infty$) for $n < cst. N_{Ch} \sim \kappa^{-3/2}$

By ineqs. à la Hardy-Littlewood, Sobolev,

$$\|\Psi\|_{1/2} \leq \left(1 - cst. \kappa n^{\frac{2}{3}}\right)^{-1} \mathcal{H}(\bar{\Psi}, \Psi)$$

conserved!

$\Rightarrow$ Upper bound on $\|\Psi(t)\|_{1/2}$ in terms of $\mathcal{H}(\bar{\Psi}(0), \Psi(0))$, $\forall t$

3. "Star formation"

If $\mathcal{H}(\bar{\Psi}(0), \Psi(0)) < nm$

(nm is threshold energy for n far separated electrons of mass m)

then

$$\|\Psi(t)\|_{L^p} \not\to 0, \text{ as } t \to \infty,$$

for arb. $p \geq 2$.

$\longrightarrow$ "binding"

($\nearrow$ E. Lenzmann, loc. cit.)

4. Finite-time blow-up

$\rho_{\Psi(0)}(x)$ rad. symmetric

$$\mathcal{H}_H(\bar{\Psi}(0), \Psi(0)) < 0$$

exchange term neglected!

Then $T < \infty$,

(7) $$\lim_{t \nearrow T} \|\Psi(t)\|_{1/2} = \infty,$$

$$\liminf_{t \nearrow T} \int_{|x| \leq R} \rho_{\Psi(t)}(x)\, dx \geq \frac{cst.}{\varkappa^{3/2}}$$

for every $R > 0$.

Blow-up profile?

Sketch of proof of (7):

We will use a virial argument:

$$Q := x \cdot \sqrt{p^2 + m^2}\, x > 0$$

$(p = -i\nabla)$

$$\Psi(0) = \left(\psi_k(0)\right)_{k=1}^{n}, \; \psi_k(0) \in C_0^\infty(\mathbb{R}^3),$$

$$\forall\, k = 1, \dots, n \quad \left(n > \text{const.}\, \kappa^{-3/2}\right)$$

$$\mathcal{H}_H\left(\overline{\Psi}(0), \Psi(0)\right) = -\mathcal{E}_0 \overset{!}{<} 0$$

$\Psi(t)$ solu. of Hartree Eq.

$$i\partial_t \Psi(t) = \sqrt{-\Delta + m^2}\, \Psi(t) + V_t \Psi(t),$$

$$V_t := -\frac{\kappa}{|x|} * \rho_{\Psi(t)} : \text{spher. symm.}$$

$$q(t) := \langle \Psi(t), Q\Psi(t)\rangle > 0$$

$$A := \frac{1}{2}(x \cdot p + p \cdot x)$$

(gen. of dilatations)

(i) $\dot{q}(t) \leq 2\langle \Psi(t), A\Psi(t)\rangle + C\, \|\Psi(t)\|_2^4$

$$\overset{\shortparallel}{\|\Psi(0)\|_2^4}$$

(ii) $$\frac{d}{dt}\langle\Psi(t), A\,\Psi(t)\rangle \leq 2\mathcal{H}_H(\bar{\Psi}(t), \Psi(t$$

$$= 2\mathcal{H}_H(\bar{\Psi}(0), \Psi(0)) = -2\mathcal{E}_0$$

Integration of (ii) & (i) $\Rightarrow$

$$0 < q(t) < \underbrace{-\mathcal{E}_0 t^2 + \mathcal{A}t + \mathcal{B}}_{\text{has a zero at } t_* < \infty}$$

$\Rightarrow \quad T < \infty\,!$

(ii) is easy;

proof of (1):

$$\dot{q}(t) = i\langle\Psi(t), \underbrace{[\sqrt{p^2+m^2}, Q]}_{=-2iA}\Psi(t)\rangle$$

$$+\, i\langle\Psi(t), [V_t, Q]\Psi(t)\rangle$$

Calculating $[V_t, Q]$ and applying Newton's theorem

$$|V_t(x)| \leqslant \frac{\|\Psi(t)\|_2^2}{|x|}, \quad |\nabla V_t(x)| \leqslant \frac{\|\Psi(t)\|_2^2}{|x|^2}$$

we find that

$$\|[V_t, Q]\| \leqslant C'\left(\|\nabla(x^2 V_t)\|_\infty + \||x| V_t\|_\infty\right)$$
$$\leqslant C'' \|\Psi(t)\|_2^2$$

■

---

(1) Inertially moving stars as trav. solitary waves

(2) Stars moving in ext. grav. potential V for Boson Stars

3. Newtonian point particles as "solitons" of cont. theories

Consider, e.g., Hartree Eq. as q.t. model of cont. medium:

$$
\begin{aligned}
i\hbar\partial_t \psi_t(x) &= \big(T+V(\varepsilon x)\big)\psi_t(x) \\
&\quad - g\left(|\psi_t|^2 * \phi\right)(x)\,\psi_t(x)
\end{aligned}
\tag{5}
$$

$$T = -\frac{\hbar^2\Delta}{2m},\ \sqrt{-\hbar^2\Delta+m^2},\ \dots$$

V: e.g., grav. pot. of central "star", $\|V\|_\infty < \infty$.

$$\phi(x) = \frac{1}{|x|},\ \frac{e^{-\mu|x|}}{|x|},\ \dots$$

$$\|\psi_t\|_2^2 =: \nu = O(1),\ g > 0.$$

(Model of a "Boson star.")

Hamilton functional:

$$\mathcal{H}(\bar{\psi},\psi;\varepsilon) = \int dx \left\{ \overline{\psi(x)}(T\psi)(x) + \left[V(\varepsilon x) - g\int dy\, |\psi(y)|^2 \phi(y-x)\right] |\psi(x)|^2 \right\}$$

Conservation laws:

- $\mathcal{N}(\bar{\psi},\psi) := \int |\psi(x)|^2 dx \leftrightarrow$ gauge inv.

For $\varepsilon = 0$,

- $\mathcal{P}(\bar{\psi},\psi) := -i\hbar \int (\bar{\psi}\nabla\psi)(x)\, dx$

$\leftrightarrow$ translation inv.

Consider "energy funct."

$$\mathcal{E}_v(\psi) := \mathcal{H}(\bar{\psi},\psi;\varepsilon=0) + v\cdot\mathcal{P}(\bar{\psi},\psi)$$

$v \in \mathbb{R}^3$: C-of-M velocity.

(For pseudo-relat. T, $|v| < 1$)

Var. problem: Construct minimizer, $\varphi_{v,\mu}$, for $\mathcal{E}_v$, with $\|\varphi_{v,\mu}\|_2^2 = \nu(\mu)$.

Solves eq.

$$T\varphi_{v,\mu} + iv\cdot\nabla\varphi_{v,\mu} - g(|\varphi_{v,\mu}|^2 * \phi)\varphi_{v,\mu} + \mu\varphi_{v,\mu} = 0$$

(Subaddit. + concentr.-comp.)

Then

$$(6) \qquad \psi_t(x) := e^{i\theta(t)}\varphi_{v,\mu}(x-q-vt)$$

solves Hartree eq. (5): solitary wave sol. describes giant "molecule",

e.g., a "Boson star", of bound matter travelling inertially w. velocity v.
(For $T = -\frac{\hbar^2 \Delta}{2m}$, φ_v obtained from φ_0 by Galilei boost.)

Solu. (6) of (5) depends on 8 parameters:

$$\xi := (q, v, \mu, \theta) \in S \subset \mathbb{R}^8.$$

ξ: coos. of point in 8-dim. surface in $\Gamma = H^1(\mathbb{R}^3)$:

$$\mathcal{M}_S := \left\{ \varphi_{v,\mu}(\cdot - q) \,\middle|\, \varphi_{v,\mu}(0) = e^{i\theta}, \; \|\varphi_{v,\mu}\|_2^2 = \nu(\mu) \right\}$$

"soliton manifold"

$\omega\big|_{\mathcal{M}_s}$: non-degenerate

Let ψ_t be solu. of (5) for $\varepsilon > 0$, w. $\mathrm{dist}(\psi_0, \mathcal{M}_s) < O(\varepsilon)$

Let φ_{ξ_t} be "skew-orth." proj. of ψ_t onto $\mathcal{M}_s$

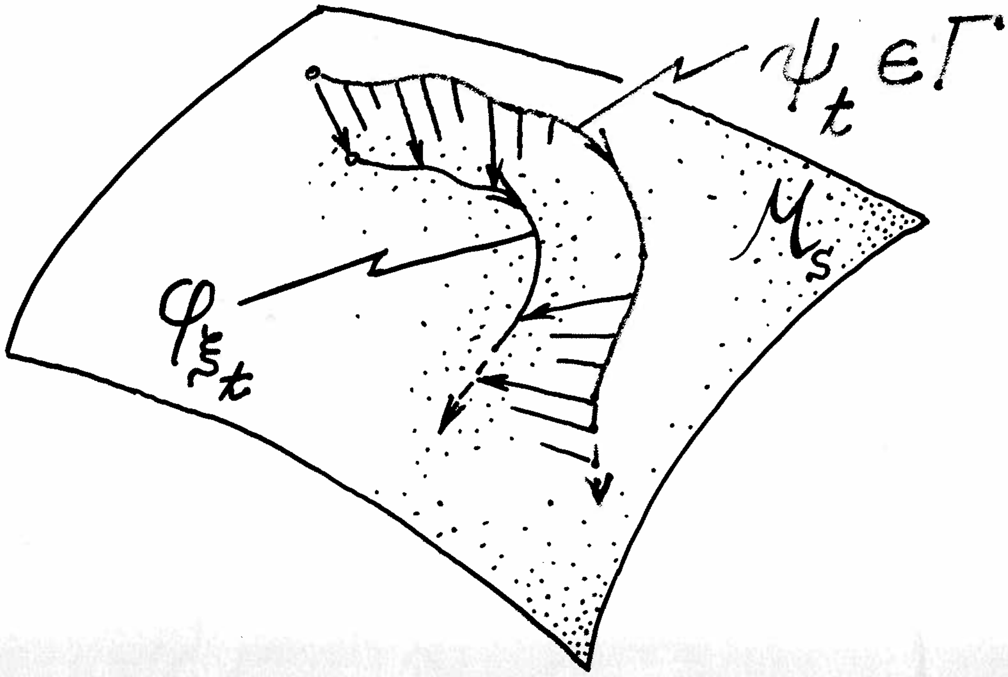
<u>Theorem.</u> For $|t| < O(\varepsilon^{-1\ldots})$,

$\mathrm{dist}(\psi_t, \mathcal{M}_s) \sim O(\varepsilon) \Rightarrow$

φ_{ξ_t} well def. & unique;

(N)
$$\boxed{\begin{aligned} \dot{q}_t &= v_t + O(\varepsilon^2) \\ \gamma \dot{v}_t &= -\varepsilon (\nabla V)(\varepsilon q_t) + O(\varepsilon^2) \end{aligned}}$$

$\dot{\mu} = O(\varepsilon^2)$, $\dot{\theta} = \mu - V(\varepsilon q) + O(\varepsilon^2)$



<u>Interpretation</u>: In an ext. pot. $V(\varepsilon\,\cdot)$, center of mass, q_t, of "molecule" in state ψ_t is solution of Newton's eqs. of motion for point particle in pot. $V(\varepsilon\,\cdot)$ with friction force $\sim O(\varepsilon^2)$

$\downarrow$

structure formation

($\nearrow$ e.g. T. Tao)

Further results.

(1) Motion of several interacting solitons (J.F.; A-S, F, S)

(2) Asy. stability & scattering (R, S, S; G.P.; E.L.)

(3) Ext. to Hartree-Fock & BHF $\longrightarrow$

dynamical approach to Chandrasekhar limit for white dwarfs;

? BCS pairing of neutrons in groundstate of neutron star; ...

2-body problem: Krieger, Martel, Raphaël

"A moving body will come to rest as soon as the force pushing it no longer acts on it in the manner necessary for its propulsion."

(Aristotle)

Aristotle thought of a mechanics of *motion with friction*, whose fundamental law was that the *velocity of a moving body is proportional to the force pushing it*. Everyday experience compels one to think that his point of view has its merits.

The *basic problem* discussed in this lecture is to understand how Aristotle's mechanics can be derived from *Newton's* in appropriate regimes.

This is a report on joint work with
De Roeck, Gang, Pizzo, Soffer, Sigal et al.

CONTENTS

1. Introduction
2. A "simple" model
3. Diffusion & friction
4. A theorem deserving this name
5. Epilogue

1. INTRODUCTION

Fundamental - Emergent
Dynamics

Hamiltonian -	Dissipative -
Celestial Mech.	Friction, diffusion, BM
Schrödinger Eq.	Lindblad Eq.

. . . . .

▸ Dynamics of point particles coupled to dispersive medium:
e.m. field; gas; BEC →
Cerenkov rad., (Q)BM, ...

2. A "SIMPLE" MODEL

Point particle coupled to Bose gas exhibiting BEC.

Hamiltonian of particle.

$$\mathcal{H}_p(P,X) = \frac{P^2}{2M} + V(X), \quad X \in \mathbb{R}^3.$$

$$V(X) \overset{e.g.}{=} F \cdot X \quad (\text{cst. force})$$

Hamiltonian of Bose gas,

(in mean-field limit)

Ginzburg-Landau descr.

$\psi(x) \in \mathbb{C}, x \in \mathbb{R}^3$: order param.

($\nearrow$ annihilation op.)

$|\psi(x)|^2$: density of gas at x.

$\rho > 0$: mean density of gas

$$\mathcal{H}_{BG}(\overline{\psi}, \psi) := \int dx \left\{ \frac{1}{2m} |\nabla \psi(x)|^2 + \right.$$

$$\left. \frac{\lambda}{2} \int dy \left(|\psi(x)|^2 - \rho\right) \Phi(x-y) \left(|\psi(y)|^2 - \rho\right) \right\}$$

Φ : short range, pos. type; $\lambda \geq 0$.

Phase space $\Gamma := H^1(\mathbb{R}^3)$;

Poisson brackets:

$$\{\overset{(-)}{\psi}(x), \overset{(-)}{\psi}(y)\} = 0, \{\psi(x), \overline{\psi}(y)\} = i\delta(x-y)$$

(QM: $\{\cdot,\cdot\} \mapsto -i\varkappa[\cdot,\cdot]$!)

Symmetry: $\overset{(-)}{\psi}(x) \mapsto e^{\overset{(-)}{+} i\theta} \overset{(-)}{\psi}(x)$

— breaking:

$$\psi(x) = \sqrt{\rho} + \beta(x), \text{ with}$$

$$\beta(x) \to 0, \quad |x| \to \infty.$$

→ BEC ; Goldstone modes
?

Simplification: Bogoliubov lim

$\lambda \to 0, \rho \to \infty, \quad \lambda \cdot \rho = cst. =: \varkappa$

Then

$$\mathcal{H}_{BG}(\bar\beta, \beta) = \int dx \left\{ \frac{1}{2m} |\nabla \beta(x)|^2 + \right.$$

$$\left. + 2\varkappa \int dy \, \mathrm{Re}\beta(x) \Phi(x-y) \mathrm{Re}\beta(y) \right\}$$

$$= \int dx \left\{ \frac{1}{2m} \left[(\nabla\pi)^2(x) + (\nabla\varphi)^2(x) \right] + \right.$$

$$\left. + 2\varkappa \int dy \, \varphi(x) \Phi(x-y) \varphi(y) \right\}$$

where $\beta(x) = \varphi(x) + i\pi(x)$

→ Eqs. of motion:

$$\left.\begin{aligned} \ddot\varphi = \frac{-1}{2m} \Delta \dot\pi &= -\frac{1}{4m^2} \Delta^2 \varphi \\ &+ \frac{4\varkappa}{m} (\Delta\Phi) * \varphi \end{aligned}\right\} \text{"Wave Eq."}$$

→ Dispersion law (by Fourier tr.)

$$\Omega(k) = |k| \sqrt{\left(\frac{k}{2m}\right)^2 + \frac{4\varkappa}{m} \hat\Phi(k)}$$

$$\simeq |k| \underbrace{2\sqrt{\varkappa \cdot \hat\Phi(0)/m}}$$

$\equiv v_*$ (prop. speed of Goldstone m's)

$$\mathcal{H}_{BG} \sim \int dk \; \bar{\hat\beta}(k) \Omega(k) \hat\beta(k)$$

Interaction particle-Bose gas.

$$\mathcal{H}_I(X; \bar\beta, \beta) = g \int dx \, W(X-x) \left(|\psi(x)|^2 - \rho \right)$$

$$= g \int dx \, W(X-x) \left[\cancel{|\beta(x)|^2} + 2\sqrt{\rho}\, \varphi(x) \right]$$

$\rho \to \infty, \; g \to 0, \quad 2g\sqrt{\rho} = cst. =: \nu$

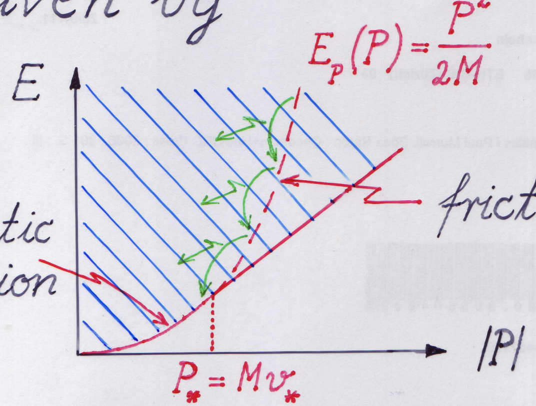
W: spher. symm., short range

$$\mathcal{H} := \mathcal{H}_P + \mathcal{H}_{BG} + \mathcal{H}_I .$$

3. DIFFUSION & FRICTION

Dynamics of particle, I – QM

(1) First, set $\mathcal{H}_I = 0$: Joint energy-mom. spect., $F=0$, given by



Turn on $\mathcal{H}_I$: 1-P states with $|P| > P_* \longrightarrow$ resonances decaying by emission of Cerenkov rad. of Goldstone modes untill $|P| \leq P_*$; ($\nearrow$ DeR – F – P).

(2) $T > 0$. Expected behavior

$$\langle X_t \rangle_T \underset{t\to\infty}{\sim} v_F \cdot t\,,$$

$$\langle (X_t - v_F \cdot t)^2 \rangle_T \underset{t\to\infty}{\sim} D \cdot t\,,$$

with

$$\boxed{\partial v_F / \partial F \Big|_{F=0} = \frac{1}{k_B T} D} \quad \text{(Einstein)}$$

(Proven for simplified models by DeR – F – S; first ever der. of QBM!)

Dynamics of particle, II – CM

Eqs. of motion at $T=0$ are:

$$M\dot{X}_t = P_t\,,$$

$$\dot{P}_t = -\nabla V(X_t) - \nu \int dx\, \nabla W(X_t - x)\, \varphi_t(x),$$

$$i\dot{\beta}_t = -\frac{\Delta}{2m}\beta_t + 2\varkappa\Phi * \varphi_t + \nu W^{X_t},$$

where $\beta = \varphi + i\pi$, $W^X(x) := W(X-x)$.

Some special solutions.

(1) Traveling waves for $F=0$:

$$\left.\begin{array}{l} P_t = Mv, \quad X_t = vt + X_0 \\ \beta_t(x) = \gamma_v(x - vt - X_0) \end{array}\right\} (TW)$$

$$\Rightarrow -iv\cdot\nabla\gamma_v = -\frac{\Delta}{2m}\gamma_v + 2\varkappa\Phi * \mathrm{Re}\,\gamma_v + \nu W$$

F.T. $\Big\{$ This eq. has regular real solu. if $|v| < 2\sqrt{\varkappa\hat{\Phi}(mv)/m} < v_*$

Hence, in an interact. env. ($\varkappa > 0$), you don't slow down if your speed is well below speed of sound of env.!

If $|v| > v_*$: $\hat{\gamma}_v$ is singular, and

$$\dot{P} = \nu\,\mathrm{Re}\int dx\, W\nabla\gamma_v \neq 0$$

with $v\cdot\dot{P} < 0 \Rightarrow$ friction!

You slow down if you try to run faster than your env.!

In a boring envir., i.e., $\varkappa = 0$ hence $v_* = 0$, you slow down until $v = 0$ (complete rest).

(2) Forced traveling waves, $F \neq 0$:

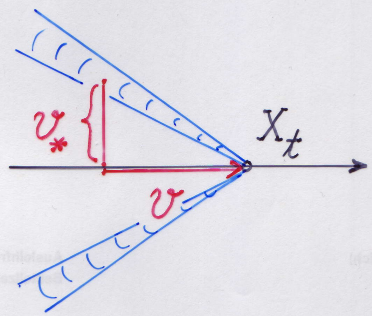
$$M\dot{X}_t = P_t, \quad \dot{P}_t = F + \nu\,\mathrm{Re}\int dx\, W\nabla\gamma_v$$

$$\dot{P}_t \overset{!}{=} 0$$

γ_v as above.

<u>Theorem</u>. If W is smooth then $\exists\, F_{max} < \infty$ s.t., for $|F| < F_{max}$, there are 2 TW solutions prop. with speeds v_F^- (stable), $v_F^+ > v_F^-$ (unstable). If $|F| > F_{max}$ $\nexists$ stat. TW solution.

Solution looks like



"Fermi's Golden Rule" + stationary phase

Thus, if you want to move at a speed > speed of sound of environment have yourself pushed gently by ext. force (e.g., collaborator).

4. A THEOREM DESERVING THIS NAME

We set $\varkappa = 0$ (ideal Bose gas) and assume that W is

- smooth & of rapid decay
- spherically symmetric
- $\hat{W}(0) \neq 0$;

$V(X) \equiv 0$. $\rightarrow$ "B-Model"

Eqs. of motion are:

$$\dot{X}_t = P_t/M,\ \dot{P}_t = -\nu\int dx\, W(X_t - x)\nabla \mathrm{Re}\beta_t(x$$

$$i\dot{\beta}_t = -\frac{\Delta}{2m}\beta_t + \nu W^{X_t}$$

Theorem. Under ass. stated above, given $\delta \in (0, \delta_*)$, where $\delta_* \approx .66$, $\exists\ \varepsilon = \varepsilon(\delta) > 0$ s.t. if $\|(1+|x|^2)^{3/2}\beta_0\|_2 < \varepsilon$, $|P_0| < \varepsilon$

then $|P_t| \leq cst.\, t^{-\frac{1}{2}-\delta}$, as $t\to\infty$

$$\lim_{t\to\infty}\|\beta_t - 2m\nu\Delta^{-1}W^{X_t}\|_\infty = 0;$$

if $\delta > \frac{1}{2}$ $\quad X_t \to X_\infty,\ |X_\infty| < \infty$.

Remark. Not known whether δ depends on initial cond.

General idea of proof:

Solve eq. for β_t & plug solu. into eq. for $P_t \Rightarrow$

$$\dot{P}_t = \underbrace{\underset{\text{lin.}}{L_1(P.)(t)} + \underset{\text{lin.}}{L_2(P.)(t)} + \underset{\text{non-lin.}}{N(P.)(t)}}_{\text{friction force w. memory}}$$

Here

$$L_1(P.)(t) = -\int_0^t ds\, f(t-s)P_s,$$

$$f(s) := cst.\, \mathrm{Re}\langle W, e^{i(\Delta/2m)s}W\rangle$$

Let K_t be scalar solu. of

$$\dot{K}_t = L_1(K.)(t),\quad K_0 = 1.$$

Then

$$(1)\quad P_t = K_t P_0 + \int_0^t ds\, K_{t-s}\left[L_2(P.)(s) + N(P.)(s)\right]$$

and

$$(2)\quad P_t = P_0 + \int_0^t ds \left[L_1(P.)(s) + L_2(P.)(s) + N(P.)(s) \right]$$

Consider: $(1) - K_t * (2)$;

use props. of K_t to solve an appropriate fixed-point problem in a weighted Banach space $\rightarrow$ P_t, exp. δ

5. EPILOGUE

(✓) • Analogous results for "E-model" ($\kappa > 0$).

Subsonic (✓) and supersonic (✓) motions of particle in E-model (↗ with Gang Zhou et al.)

- Derivation of mean-field limits (in thermo-dynamic limit; ↗ D-F-P-P)
- Forced motion (V≠0: Relaxation to an equilibrium point; or F≠0: Instabilities!)
- N-particle problems, with N ≥ 2:

  binding, binary collapse, etc.
- Gases of particles suspended in a medium (e.g., a Bose gas) at positive temperature: Derivation of Boltzmann Equation in the Grad limit (↗ O. E. Lanford), solutions and approach to equilibrium (✓ , thanks to Gang Zhou)

Remarks on Effective Dynamics in Quantum Systems

We conclude with some remarks on the effective dynamics of a quantum system, S, e.g., a quantum particle hopping on a lattice, coupled to a dispersive environment, E, (an ideal quantum gas) in thermal equilibrium at T> 0, or in its groundstate (T=0), after tracing out the degrees of freedom of E.- This is an area where much progress has been made recently. I would like to propose a list of problems that, I believe, are of interest to mathematicians.

Effective ("emergent") dynamics

Couple S to "environment" E:

$$S \vee E\,,\quad \mathcal{B}_{S\vee E} = \mathcal{B}_S \otimes \mathcal{B}_E\,,$$

$\mathcal{B}_S \overset{e.g.}{\simeq} B(\mathcal{H}_S)$. State ω on $\mathcal{B}_{S\vee E}$,

time evol. $\{\alpha_{t,s}\}$.

$$tr(P_S\, a) := \omega(a\otimes \mathbb{1})\,,\quad a\in\mathcal{B}_S\,.$$

Eff. time evol. of S:

$$tr((Z_{t,t_0} P_S)\, a) := \omega(\alpha_{t,t_0}(a\otimes\mathbb{1}))\,,$$

$a\in\mathcal{B}_S$. In gen., $Z_{t,s}\circ Z_{s,u} \neq Z_{t,u}$!

Quantum Markovian dyn.:

$$Z_{t,s} \approx T_{t,s}\,,\quad T_{t,s}: \mathcal{J}_1^+(\mathcal{H}_S) \circlearrowleft\,, \;\&$$

$$T_{t,s}\circ T_{s,u} = T_{t,u}\,,\quad \forall\, t \geq s \geq u\,. \quad (CK)$$

Theorem (Lindblad)

If $(T_{t,s})$ satisfies (CK) & completely positive then it is generated by "Lindbladians" $(\mathcal{L}_t)_{t\in\mathbb{R}}$, where

$$\mathcal{L}_t(P_s) = -i[H_t, P_s] + \Psi_t(P_s) \leftarrow \text{gain}$$
$$-\frac{1}{2}\{\Psi_t(\mathbb{1}), P_s\} \leftarrow \text{loss}$$

with Ψ_t completely positive.

Problem 5. (hard analysis)

Explore props. of $T_{t,s}$ $(t \to \infty)$; expand $Z_{t,s}$ around some $T_{t,s}$, uniformly in t. $\to$ QBM

Exc. area with new results!

QM models of a particle inter-aciting w. dispersive medium

(i) $H_t = -\Delta + \phi_\omega(x,t)$ on $\ell^2(\mathbb{Z}^d)$,

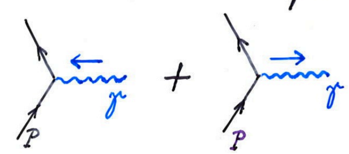
ϕ_ω Markov random field,

$$\overline{\phi_\omega(x_1,t_1)\,\phi_\omega(x_2,t_2)} \propto G(x_1-x_2)\,e^{-\frac{|t_1-t_2|}{\tau}}$$

$$tr\left(P_0\left(x(t)-x(0)\right)^2\right) \sim Dt$$

($\to$ Erikhman & Ovchinnikov, Kang & Schenker)

(ii) $H = -\Delta + W + H_{ph}$

W : 

H_{ph} : free photon dynamics

21

Initial state of photons, γ:

- vacuum $(T=0)$
- thermal equ. $(T>0)$ - Planck

—"— of particle: P_0 loc. near 0.

$\underline{T=0}$: Friction by emission of Cerenkov rad. ($\nearrow$ Gang Zhou); "infra-particle" dyn. ...

$\underline{T>0}$: Diffusion, QBM

$$\langle (x(t)-x(0))^2 \rangle \sim D t \quad (d \geq 4)$$

($\nearrow$ De Roeck ...)

Einstein rel.: $\frac{\partial v_F}{\partial F} = \beta D$

F: const. ext. force

($\nearrow$ PhD Schnelli). Etc.

Another class of problems concerns the derivation of an effective unitary (or Hamiltonian) dynamics, accurate in a suitable limiting regime, for a small system, such as a tracer particle, coupled to a medium with infinitely many degrees of freedom that "ignores" the degrees of freedom of the medium. (*Example*: Electron coupled to quantized em field, under the influence of a slowly varying external potential: ➚ BCFFS – Justification of notion of "closed phys. system".)

Everything else, next time – Thank you for your attention! I am happy to take some questions.

Examples of Hamiltonian Evolution Equations

- Euler Equations (& “quantization” ➔ vortex dynamics)
- Vlasov Eq. (& its “quantization” ➔ point-particle mechanics) Maxwell-Vlasov, etc.
- Non-linear Schrödinger– and Hartree Eqs. (non-focusing & *focusing*: e.g., boson stars, structure formation; soliton dynamics, KAM theorems for soliton dynamics (?), etc.)
- Hartree-Fock- and Bogoliubov-Hartree-Fock Eqs. (e.g., collapse of white dwarfs – collapse profile; atomic and molecular physics, superconductivity, etc.)
- Maxwell-axion dynamics (➔ growth of cosmic magnetic fields)
- Coupled particle-wave dynamics, with wave medium = Bose gas, or electromagnetic field – Cherenkov radiation, Hamiltonian friction, etc.; (Gang’s talks).